

Multi-loop correlators for rational theories of 2D gravity from the generalized Kontsevich models

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Abstract

We introduce a parametrization of the coupling constant space of the generalized Kontsevich models in terms of a set of moments equivalent to those introduced recently in the context of topological gravity. For the simplest generalization of the Kontsevich model we express the moments as elementary functions of the susceptibilities and the eigenvalues of the external field. We furthermore use the moment technique to derive a closed expression for the genus zero multi-loop correlators for $(3, 3m - 1)$ and $(3, 3m - 2)$ rational matter fields coupled to gravity. We comment on the relation between the two-matrix model and the generalized Kontsevich models.

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1 Introduction

Several types of matrix models have proven to have a singularity structure to which a scaling behaviour characteristic of two-dimensional conformal field theories coupled to gravity can be associated. The simplest example is the generic 1-matrix model which possesses singular points which, when approached by the double scaling procedure [1], give rise to the scaling behaviour characteristic of $(2, 2m - 1)$ conformal matter coupled to gravity [2]. However, the generic 1-matrix model is not the most economic way of studying the series of $(2, 2m - 1)$ theories because in the vicinity of any of its m 'th multi-critical points there will be subleading singularities present. The most economic way of studying the interaction of $(2, 2m - 1)$ matter with gravity is by using the Kontsevich model [3, 4]. In the parameter space of this model one recovers all the m 'th multi-critical regimes of the 1-matrix model but without the presence of any subleading singularities.

Recently it has been shown that the two-matrix model possesses critical points capable of describing the interaction of any (p, q) rational matter field with 2-dimensional quantum gravity [5]. As in the previous case, due to the presence of subleading singularities, the generic 2-matrix model is not the optimal tool for studying these interactions. The optimal line of action would be to consider a model which possessed singular points of the same type as those of the two-matrix model but were deprived of any subleading singularities. Although the situation is not completely as clear as in the 1-matrix case, the generalized Kontsevich models seem to be the models we are looking for [3, 6, 7, 8]. The partition functions of these models read in the normalization of reference [7]

$$Z_p^N(\Lambda) = \frac{\int dM \exp \left(\frac{i^{p^2+1}}{2(p+1)} \text{tr} \left[(M + (-i)^{p+1} \Lambda)^{p+1} \right]_{>lin} \right)}{\int dM \exp \left(-\frac{1}{4} \text{tr} \left[\sum_{k=0}^{p-1} M \Lambda^k M \Lambda^{p-1-k} \right] \right)} \quad (1.1)$$

where the integration is over $N \times N$ hermitian matrices and where Λ is an external field. The subscript $> lin$ means that only terms of degree larger than or equal to two in M should be taken into account. The usual Kontsevich model is recovered for $p = 2$. All matrix integrals of the type (1.1) have an expansion in powers of the traces, $\text{tr} \Lambda^{-n}$, but $Z_p^N(\Lambda)$ is independent of $\text{tr} \Lambda^{-np}$ [3, 7]. Expressed in terms of the remaining traces $Z_p^N(\Lambda)$ is known to fulfill a set of W_p constraints or equivalently to be a τ -function of the p^{th} reduction of the KP hierarchy, the KdV $_p$ hierarchy, and fulfill the string equation $L_{-1} Z_p^N(\Lambda) = 0$ [7, 8]. Hence $Z_p^N(\Lambda)$ should be capable of describing the coupling of rational matter fields of the type $(p, pm - 1), \dots (p, pm - (p - 1))$ to two-dimensional quantum gravity.

In the present paper we will study the $p = 3$ version of the model (1.1) with the aim of extracting information about the interaction of $(3, 3m - 1)$ and $(3, 3m - 2)$ matter

fields with gravity. In the case of the ordinary Kontsevich model as well as in the case of the generic 1-matrix model it proved convenient to parametrize the coupling constant space by a set of so-called moment variables [7, 9]. After having fixed the notation in section 2 we will in section 3 introduce the appropriate moment variables for the $p = 3$ version of (1.1) and sketch how the idea can be generalized to the generic case. Our moment variables are exactly equivalent to those introduced in reference [10] in the context of topological gravity. However, in the matrix model framework one can express the moments explicitly in terms of elementary functions of the susceptibilities and the eigenvalues of the external matrix. We hence obtain an expression for the genus zero contribution to the free energy of the $p = 3$ model where the singularities are clearly exposed. In section 4 we use the moment technique to study the macroscopic loops of $(3, 3m-1)$ and $(3, 3m-2)$ matter fields coupled to quantum gravity. We derive a closed expression for the n -loop correlator at genus zero thereby generalizing the expression obtained earlier for the $(2, 2m-1)$ case in references [11, 12]. Our results allow us to extract several characteristics of the multi-loop correlators of the generic (p, q) -model.

In section 5 we comment on the exact relation between the two-matrix model and the generalized Kontsevich models as well as the relation of the present formalism to that of strings with discrete target space [13].

2 The model

The model that we will consider is the $p = 3$ version of the partition function given in (1.1)

$$Z_N(\Lambda) = e^{F_N(\Lambda)} = \int_{N \times N} d\mu(M) \exp \left\{ -\frac{1}{2} \text{tr} \left(M^3 \Lambda + \frac{1}{4} M^4 \right) \right\} \quad (2.1)$$

where the measure reads

$$\mu(M) = \frac{dM \exp \left(-\frac{1}{2} \text{tr} \left[\Lambda^2 M^2 + \frac{1}{2} \Lambda M \Lambda M \right] \right)}{\int dM \exp \left(-\frac{1}{2} \text{tr} \left[\Lambda^2 M^2 + \frac{1}{2} \Lambda M \Lambda M \right] \right)}. \quad (2.2)$$

We introduce time variables $\{\theta_k, \tilde{\theta}_k\}$ for the model by

$$\theta_k = \frac{1}{3k+1} \text{tr} \Lambda^{-3k-1}, \quad \tilde{\theta}_k = \frac{1}{3k+2} \text{tr} \Lambda^{-3k-2}, \quad k \geq 0. \quad (2.3)$$

Then the fact that $Z_N(\Lambda)$ is a τ -function of the kdV₃ hierarchy can be expressed as

$$\frac{\partial Q}{\partial \theta_s} = [Q_+^{(3s+1)/3}, Q], \quad \frac{\partial Q}{\partial \tilde{\theta}_s} = [Q_+^{(3s+2)/3}, Q], \quad (2.4)$$

where

$$Q = \left(\frac{\partial}{\partial \theta_0} \right)^3 + \frac{3}{2} \left\{ u_1, \frac{\partial}{\partial \theta_0} \right\} + 3u_2 \quad (2.5)$$

$$u_1 = \frac{\partial^2 F}{\partial \theta_0^2}, \quad u_2 = \frac{1}{2} \frac{\partial^2 F}{\partial \theta_0 \partial \tilde{\theta}_0} \quad (2.6)$$

and the constraint $L_{-1}Z_N(\Lambda)=0$ implies that

$$\frac{\partial u_1}{\partial \theta_0} = \frac{2}{3} \sum_{s \geq 0} (3s+4) \theta_{s+1} \frac{\partial u_1}{\partial \theta_s} + \frac{2}{3} \sum_{s \geq 0} (3s+5) \tilde{\theta}_{s+1} \frac{\partial u_1}{\partial \tilde{\theta}_s} \quad (2.7)$$

$$\frac{\partial u_2}{\partial \theta_0} = \frac{2}{3} \sum_{s \geq 0} (3s+4) \theta_{s+1} \frac{\partial u_2}{\partial \theta_s} + \frac{2}{3} \sum_{s \geq 0} (3s+5) \tilde{\theta}_{s+1} \frac{\partial u_2}{\partial \tilde{\theta}_s} + \frac{2}{3}. \quad (2.8)$$

This information allows us to solve the model to any order in $1/N^2$. The time variables $\{\theta_k, \tilde{\theta}_k\}$ are related to the $t_{n,m}$, $n = 0, 1$; $m \geq 0$ used by Witten [4] in the context of topological gravity by

$$\theta_k = t_{0,k} \frac{\rho^{3k+1}}{(3k+1)!!!} \left(\frac{\sqrt{3}}{i} \right)^k, \quad \tilde{\theta}_k = t_{1,k} \frac{\rho^{3k+2}}{(3k+2)!!!} \left(\frac{\sqrt{3}}{i} \right)^{k+1}, \quad (2.9)$$

$$\rho^4 = \frac{3^{3/2}}{2i}, \quad (3k+m)!!! = (3k+m)(3(k-1)+m) \dots m, \quad m = 1, 2.$$

3 The solution

Let us introduce the following notation for the negative part of the pseudo differential operators entering the equations (2.4)

$$Q_-^{(3s+1)/3} \equiv P_s \partial^{-1} + (Q_s - \frac{1}{2} P'_s) \partial^{-2} + \mathcal{O}(\partial^{-3}), \quad (3.1)$$

$$Q_-^{(3s+2)/3} \equiv \tilde{P}_s \partial^{-1} + (\tilde{Q}_s - \frac{1}{2} \tilde{P}'_s) \partial^{-2} + \mathcal{O}(\partial^{-3}) \quad (3.2)$$

where $\partial = \frac{\partial}{\partial \theta_0}$ and primes refer to differentiation with respect to θ_0 . Then the flow equations for u_1 and u_2 can be written as

$$\frac{\partial u_1}{\partial \theta_s} = P'_s, \quad \frac{\partial u_1}{\partial \tilde{\theta}_s} = \tilde{P}'_s \quad (3.3)$$

$$\frac{\partial u_2}{\partial \theta_s} = Q'_s, \quad \frac{\partial u_2}{\partial \tilde{\theta}_s} = \tilde{Q}'_s. \quad (3.4)$$

The functions $\{P_i, \tilde{P}_i, Q_i, \tilde{Q}_i\}$ are polynomials in u_1 and u_2 and the derivatives of these and determined by

$$P_0 = u_1, \quad Q_0 = u_2, \quad (3.5)$$

$$\tilde{P}_0 = 2u_2, \quad \tilde{Q}_0 = -\frac{1}{6} u_1'' - u_1^2 \quad (3.6)$$

plus the following set of recursion relations which can be derived in the standard way

$$P'_{s+1} = \frac{2}{3} Q_s^{(3)} + 2u_1 Q'_s + u'_1 Q_s + 2u'_2 P_s + 3P'_s u_2, \quad (3.7)$$

$$Q'_{s+1} = -\frac{1}{18} P_s^{(5)} - \frac{5}{6} u_1 P_s^{(3)} - \frac{1}{6} u_1^{(3)} P_s - \frac{5}{4} u'_1 P''_s - \frac{3}{4} u''_1 P'_s + 3u_2 Q'_s + u'_2 Q_s - 2u_1^2 P'_s - 2u_1 u'_1 P_s. \quad (3.8)$$

Let us consider the planar limit $N \rightarrow \infty$. In this limit we can neglect all higher derivatives in the recursion relations. It is possible to show that under these circumstances the polynomials take the following general form

$$P_k = (3k+1)!!! \sum_{j=0}^{[k/2]} \frac{(3j-1)!!!(-1)^j}{(k-2j)!(3j+1)!} u_1^{3j+1} u_2^{k-2j}, \quad (3.9)$$

$$Q_k = (3k+1)!!! \sum_{j=0}^{[(k+1)/2]} \frac{(3j-1)!!!(-1)^j}{(k+1-2j)!(3j)!} u_1^{3j} u_2^{k+1-2j}, \quad (3.10)$$

$$\tilde{P}_k = (3k+2)!!! \sum_{j=0}^{[(k+1)/2]} \frac{(3j-2)!!!(-1)^j}{(k+1-2j)!(3j)!} u_1^{3j} u_2^{k+1-2j}, \quad (3.11)$$

$$\tilde{Q}_k = (3k+2)!!! \sum_{j=0}^{[k/2]} \frac{(3j+1)!!!(-1)^{j+1}}{(k-2j)!(3j+2)!} u_1^{3j+2} u_2^{k-2j} \quad (3.12)$$

where $[a]$ denotes the integer part of a . These polynomials can be shown to fulfill the following relations

$$\frac{\partial P_k}{\partial u_1} = (3k+1)Q_{k-1}, \quad \frac{\partial Q_k}{\partial u_1} = -(3k+1)u_1 P_{k-1}, \quad (3.13)$$

$$\frac{\partial P_k}{\partial u_2} = (3k+1)P_{k-1}, \quad \frac{\partial Q_k}{\partial u_2} = (3k+1)Q_{k-1}. \quad (3.14)$$

Similar relations where $(3k+1)$ is replaced by $(3k+2)$ hold for the polynomials $\{\tilde{P}_k, \tilde{Q}_k\}$. Inspired by the form of the flow equations (2.7) and (2.8) let us introduce two sets of moment variables by

$$M_k = \frac{2}{3} \left\{ \sum_{s \geq -1} \theta_{s+k} P_s \frac{(3(s+k)+1)!!!}{(3s+1)!!!} + \sum_{s \geq -1} \tilde{\theta}_{s+k} \tilde{P}_s \frac{(3(s+k)+2)!!!}{(3s+2)!!!} \right\}, \quad (3.15)$$

$$J_k = \frac{2}{3} \left\{ \sum_{s \geq -1} \theta_{s+k} Q_s \frac{(3(s+k)+1)!!!}{(3s+1)!!!} + \sum_{s \geq -1} \tilde{\theta}_{s+k} \tilde{Q}_s \frac{(3(s+k)+2)!!!}{(3s+2)!!!} \right\} \quad (3.16)$$

where the polynomials with negative indices are defined by the relations (3.13) and (3.14). Let us notice that the relations (3.13) and (3.14) for the polynomials $\{P_k, \tilde{P}_k, Q_k, \tilde{Q}_k\}$ imply the following relations between the moments

$$\frac{\partial M_k}{\partial u_1} = J_{k+1}, \quad \frac{\partial J_k}{\partial u_1} = (-u_1)M_{k+1}, \quad (3.17)$$

$$\frac{\partial M_k}{\partial u_2} = M_{k+1}, \quad \frac{\partial J_k}{\partial u_2} = J_{k+1}. \quad (3.18)$$

Now we can write the flow equations (2.7) and (2.8) as

$$\frac{\partial}{\partial \theta_0} (u_1 - M_1) = 0, \quad \frac{\partial}{\partial \theta_0} (u_2 - J_1) = 0. \quad (3.19)$$

Furthermore one can show that one has in addition

$$\frac{\partial}{\partial \theta_k}(u_1 - M_1) = 0, \quad \frac{\partial}{\partial \theta_k}(u_2 - J_1) = 0, \quad k \geq 1; \quad (3.20)$$

$$\frac{\partial}{\partial \tilde{\theta}_k}(u_1 - M_1) = 0, \quad \frac{\partial}{\partial \tilde{\theta}_k}(u_2 - J_1) = 0, \quad k \geq 0. \quad (3.21)$$

This can be seen from rewritings of the following type

$$\frac{\partial u_1}{\partial \theta_k} = \frac{\partial}{\partial \theta_0} P_k = \frac{\partial P_k}{\partial u_1} \frac{\partial u_1}{\partial \theta_0} + \frac{\partial P_k}{\partial u_2} \frac{\partial u_2}{\partial \theta_0} \quad (3.22)$$

followed by application of the relations (3.13) and (3.14) as well as the constraints (2.7) and (2.8). Hence $(u_1 - M_1)$ and $(u_2 - J_1)$ must be constants and since they should vanish for $\theta_i = 0$, $\tilde{\theta}_i = 0$ one concludes that

$$u_1 = M_1, \quad u_2 = J_1. \quad (3.23)$$

These equations give us an implicit expression for F_0 . The moment variables (3.15) and (3.16) are identical to those introduced in reference [10] for the topological minimal model associated with the Lie Algebra A_2 . As in that case the description could easily be generalised to the case of A_n it is obvious that the strategy applied here for $p = 3$ version of (1.1) can easily be generalized to the $p = n$ version. For the $p = n$ version we will have $(n - 1)$ series of time variables, $(n - 1)$ susceptibilities and $(n - 1)$ relations like (2.7) and (2.8). The flow equations will be expressed in terms of $(n - 1)$ series of pseudo differential operators which have expansions like (3.1) and (3.2) where now the $(n - 1)$ first terms are of importance. Hence we are led to introduce $(n - 1)$ series of polynomials and $(n - 1)$ series of moments each moment being a sum of $(n - 1)$ terms in close analogy with (3.15) and (3.16). In reference [10] it was shown that all higher genera contributions to the free energy can be expressed entirely in terms of the moments and that for any given model, A_n , and given genus, g , only a finite number of moments appear. This result of course also appears in the matrix model framework. However, we will not enter into a discussion of this point. Let us just mention that all higher genera contributions to the free energy in the case $p = 3$ can be found by solving iteratively the genus expanded version of the flow equation

$$\frac{\partial u_1}{\partial \theta_1} = \frac{\partial}{\partial \theta_0} P_1 = \frac{\partial}{\partial \theta_0} \left(4u_1 u_2 + \frac{2}{3} \frac{\partial^2 u_2}{\partial \theta_0^2} \right). \quad (3.24)$$

In the matrix model framework it is possible to express the moment variables in terms of elementary functions of the susceptibilities and the eigenvalues of the external field. For the model (2.1) we find using the explicit expressions (3.9), (3.10), (3.11)

and (3.12) for the polynomials P_k , Q_k , \tilde{P}_k , \tilde{Q}_k and the definition (2.3) of the time variables [14]

$$M_0 = \frac{2^{4/3}}{3} u_1 \sum_k \left\{ (\lambda_k^3 - 3u_2) + [(\lambda_k^3 - 3u_2)^2 + 4u_1^3]^{1/2} \right\}^{-1/3} - \frac{2^{2/3}}{3} \sum_k \left\{ (\lambda_k^3 - 3u_2) + [(\lambda_k^3 - 3u_2)^2 + 4u_1^3]^{1/2} \right\}^{1/3} \quad (3.25)$$

$$J_0 = \frac{2^{2/3}}{3} (-u_1^2) \sum_k \left\{ (\lambda_k^3 - 3u_2) + [(\lambda_k^3 - 3u_2)^2 + 4u_1^3]^{1/2} \right\}^{-2/3} - \frac{2^{-2/3}}{3} \sum_k \left\{ (\lambda_k^3 - 3u_2) + [(\lambda_k^3 - 3u_2)^2 + 4u_1^3]^{1/2} \right\}^{2/3} \quad (3.26)$$

where $\{\lambda_k\}$ are the eigenvalues of the external field, Λ . We note that by means of the relations (3.17) and (3.18) we can express all the other moments in a similar manner. We note the presence of cubic singularities which is a well known property of $(3, 3m-1)$ and $(3, 3m-2)$ rational matter coupled to gravity. For the $p = n$ version of the Kontsevich model the moment variables depend on $(n-1)$ susceptibilities and n -root singularities are expected.

Let us integrate the equations (3.23) to obtain the genus zero contribution to the free energy, F_0 , which we will need for our considerations in the next section. Exploiting the relations (3.17) and (3.18) and assuming $\frac{dF}{d\theta_0} = \frac{dF}{d\tilde{\theta}_0} = 0$ for $\theta_0 = \tilde{\theta}_0 = 0$ it is easy to show that

$$\frac{dF_0}{d\theta_0} = \frac{3}{2} (M_0 - u_1 u_2), \quad (3.27)$$

$$\frac{dF_0}{d\tilde{\theta}_0} = 3 \left(J_0 + \frac{1}{3} u_1^3 - \frac{1}{2} u_2^2 \right) \quad (3.28)$$

and furthermore assuming $F_0 = 0$ for $\theta_0 = \tilde{\theta}_0 = 0$ one arrives at the following expression for F_0

$$F_0 = \left(\frac{3}{2} \right)^2 \left\{ \frac{1}{2} u_1 u_2^2 - u_1 J_0 - u_2 M_0 + M_{-1} + \int J_1 M_1 du_2 \right\} \quad (3.29)$$

where in the integral it is understood that J_1 and M_1 should be expressed as on the right hand side of (3.15) and (3.16) and \int is short hand notation for $\int_0^{u_2}$. This expression can of course be rewritten in the form given in [10]. We note that all terms entering (3.29) except the integral $\int J_1 M_1 du_2$ can be expressed in terms of elementary functions of u_1 , u_2 and $\{\lambda_k\}$.

4 Macroscopic Loops

In this section we shall be concerned with the calculation of macroscopic loops. By macroscopic loops we mean correlation functions of the following type

$$W^{(n)}(\pi_1, \dots, \pi_n) = \frac{d}{dV(\pi_n)} \dots \frac{d}{dV(\pi_1)} F \quad (4.1)$$

where $\frac{d}{dV(\pi)}$, the loop insertion operator, is given by

$$\frac{d}{dV(\pi)} = \sum_k \left\{ \pi^{-k-4/3} \frac{d}{d\theta_k} + \pi^{-k-5/3} \frac{d}{d\tilde{\theta}_k} \right\}. \quad (4.2)$$

Our aim will be to derive a closed expression for the genus zero contribution to the n -loop correlator, $W_0^{(n)}(\pi_1, \dots, \pi_n)$. For that purpose it is convenient to work with a slightly different version of the loop insertion operator. Using the boundary equations (3.23) it is easy to show that $\frac{d}{dV(\pi)}$ can be rewritten as

$$\frac{d}{dV(\pi)} = \frac{\partial}{\partial V(\pi)} + M_2(\pi) \hat{Q} + J_2(\pi) \hat{P} \quad (4.3)$$

where

$$\frac{\partial}{\partial V(\pi)} = \sum_k \left\{ \pi^{-k-4/3} \frac{\partial}{\partial \theta_k} + \pi^{-k-5/3} \frac{\partial}{\partial \tilde{\theta}_k} \right\} \quad (4.4)$$

and

$$\hat{P} = \Omega_1 \frac{\partial}{\partial u_1} + \Omega_2 \frac{\partial}{\partial u_2}, \quad \hat{Q} = \Omega_2 \frac{\partial}{\partial u_1} - u_1 \Omega_1 \frac{\partial}{\partial u_2}, \quad (4.5)$$

$$\Omega_1 = \frac{M_2}{(1 - J_2)^2 + u_1 M_2^2}, \quad \Omega_2 = \frac{(1 - J_2)}{(1 - J_2)^2 + u_1 M_2^2}, \quad (4.6)$$

$$M_k(\pi) = \frac{\partial M_{k-1}}{\partial V(\pi)} = M_k \Big|_{\lambda_i^3 \rightarrow \pi} \quad (4.7)$$

$$J_k(\pi) = \frac{\partial J_{k-1}}{\partial V(\pi)} = J_k \Big|_{\lambda_i^3 \rightarrow \pi} \quad (4.8)$$

where by $\lambda_i^3 \rightarrow \pi$ we mean that the functional dependence of $M_k(\pi)$ on π is like that of M_k on any of the λ_i^3 . To determine $W_0^{(1)}(\pi)$ we need only to determine $\frac{\partial F_0}{\partial V(\pi)}$ since as shown in [10] (and easily verified for the expression (3.29)) we have $\frac{\partial F_0}{\partial u_1} = \frac{\partial F_0}{\partial u_2} = 0$. With the notation of equation (4.7) and (4.8) one finds

$$W_0^{(1)}(\pi) = \left(\frac{3}{2}\right)^2 \left[-u_1 J_1(\pi) - u_2 M_1(\pi) + M_0(\pi) + \int J_1 M_2(\pi) du_2 + \int M_1 J_2(\pi) du_2 \right].$$

It is easily verified that

$$\frac{\partial}{\partial u_1} W_0^{(1)}(\pi) = \frac{\partial}{\partial u_2} W_0^{(1)}(\pi) = 0. \quad (4.9)$$

Hence the two-loop correlator reads

$$W_0^{(2)}(\pi_1, \pi_2) = \left(\frac{3}{2}\right)^2 \left[\int J_2(\pi_1) M_2(\pi_2) du_2 + \int J_2(\pi_2) M_2(\pi_1) du_2 \right]. \quad (4.10)$$

We note that as expected the two-loop correlator exhibits no explicit dependence on the time variables. Hence to find the three-loop correlator we need only to determine the effect of applying $\frac{\partial}{\partial u_1}$ and $\frac{\partial}{\partial u_2}$ to $W_0^{(2)}(\pi_1, \pi_2)$. The application of $\frac{\partial}{\partial u_2}$ is straightforward and by making use of the relations (3.17) and (3.18) one realizes that the integrals resulting from applying $\frac{\partial}{\partial u_1}$ to the integrands in (4.10) can actually be carried out by partial integration. The expression for the three-loop correlator that one arrives at is the following

$$\begin{aligned} W_0^{(3)}(\pi_1, \pi_2, \pi_3) = & \left(\frac{3}{2}\right)^2 \{ \Omega_1 J_2(\pi_1) J_2(\pi_2) J_2(\pi_3) - u_1 \Omega_2 M_2(\pi_1) M_2(\pi_2) M_2(\pi_3) \\ & + \Omega_2 [M_2(\pi_1) J_2(\pi_2) J_2(\pi_3) + \text{dis. perm.}] \\ & - u_1 \Omega_1 [M_2(\pi_1) M_2(\pi_2) J_2(\pi_3) + \text{dis. perm.}] \} \end{aligned} \quad (4.11)$$

where here and in the following by *dis. perm.* we mean permutations of π 's which give rise to truly different terms. Before proceeding to the general case let us comment on the geometrical interpretation of (4.11). For that purpose let us note that

$$\Omega_1 = \frac{3}{2} c^3 F_{000}, \quad \Omega_2 = \frac{3}{4} c^2 \tilde{c} F_{001} \quad (4.12)$$

$$(-u_1) \Omega_1 = \frac{3}{8} c \tilde{c}^2 F_{011}, \quad (-u_1) \Omega_2 = \frac{3}{16} \tilde{c}^3 F_{111} \quad (4.13)$$

where

$$F_{ijk} = \frac{d^3 F_0}{dt_{i,0} dt_{j,0} dt_{k,0}} \quad (4.14)$$

and c and \tilde{c} are given by

$$t_{0,0} = c^{-1} \theta_0, \quad t_{1,0} = \tilde{c}^{-1} \tilde{\theta}_0 \quad (4.15)$$

(cf. to equation (2.3)). Hence if we define propagators $\mathcal{P}^0(\pi)$ and $\mathcal{P}^1(\pi)$ by

$$\mathcal{P}^0(\pi) = \frac{3}{2} c J_2(\pi), \quad \mathcal{P}^1(\pi) = \frac{3}{4} \tilde{c} M_2(\pi) \quad (4.16)$$

we can write the three-loop correlator as

$$\begin{aligned} W_0^{(3)}(\pi_1, \pi_2, \pi_3) = & F_{000} \mathcal{P}^0(\pi_1) \mathcal{P}^0(\pi_2) \mathcal{P}^0(\pi_3) + F_{111} \mathcal{P}^1(\pi_1) \mathcal{P}^1(\pi_2) \mathcal{P}^1(\pi_3) \\ & + F_{011} [\mathcal{P}^0(\pi_1) \mathcal{P}^1(\pi_2) \mathcal{P}^1(\pi_3) + \text{dis. perm.}] \\ & + F_{001} [\mathcal{P}^0(\pi_1) \mathcal{P}^0(\pi_2) \mathcal{P}^1(\pi_3) + \text{dis. perm.}] \end{aligned} \quad (4.17)$$

and we see that the three-loop correlator is determined by the three-point vertices of the gravitational primary fields and that $\mathcal{P}^0(\pi)$ and $\mathcal{P}^1(\pi)$ have a natural interpretation as propagators associated with the two gravitational primary fields of the model. The formula (4.17) is a natural generalisation of the corresponding formula encountered in the case of $(p, q) = (2, 2m - 1)$ minimal models coupled to gravity [11, 12] and it is natural to expect that the three-loop correlator will have a similar structure in the generic case. For the series of rational matter fields of the type $(p, pm - 1), \dots (p, pm - (p - 1))$ coupled to gravity the propagators will exhibit p -root singularities (cf. to equations (4.7), (4.8), (3.25) and (3.26).) The decomposition of the 3-loop correlator into vertices and propagators is in perfect agreement with the Feynman rules for calculating multi-loop correlators for unitary conformal models coupled to 2D gravity obtained from the approach of strings with discrete target space [13].

Let us proceed now to the general case. To calculate the n -loop correlator ($n > 3$) we must apply the loop insertion operator $(n - 3)$ times to each of the terms in equation (4.11). The result of this process can be given in a closed form. For instance

$$\begin{aligned}
& \frac{d}{dV(\pi_{n+3})} \frac{d}{dV(\pi_{n+2})} \cdots \frac{d}{dV(\pi_4)} \{ \Omega_1 J_2(\pi_1) J_2(\pi_2) J_2(\pi_3) \} = \\
& \left\{ \sum_{k=0}^n \hat{P}^k \hat{Q}^{n-k} \Omega_1 \left[\overbrace{J_2(\pi_4) \dots J_2(\pi_{k+4})}^{k \text{ terms}} \overbrace{M_2(\pi_{k+5}) \dots M_2(\pi_{n+3})}^{(n-k) \text{ terms}} + \text{dis. perm.} \right] \right. \\
& + \sum_{k=0}^{n-1} \hat{P}^k \hat{Q}^{n-k-1} \left(u_1 \Omega_1^2 + \Omega_2^2 \right) \times \\
& \left. \left[\frac{\partial M_2(\pi_{n+3})}{\partial u_2} \overbrace{J_2(\pi_4) \dots J_2(\pi_{k+4})}^{k \text{ terms}} \overbrace{M_2(\pi_{k+5}) \dots M_2(\pi_{n+2})}^{(n-k-1) \text{ terms}} + \text{dis. perm.} \right] \right\} \times \\
& J_2(\pi_1) J_2(\pi_2) J_2(\pi_3). \tag{4.18}
\end{aligned}$$

Here $J_2(\pi_1) J_2(\pi_2) J_2(\pi_3)$ can be replaced by any function $f(u_1, u_2)$ with no explicit dependence on the time variables $\{\theta_i, \tilde{\theta}_i\}$. In particular the result immediately generalizes to the case where the loop insertion operator acts on the last term in (4.11). If one has in the first line of (4.18) in stead of a function of the type $\Omega_1 f(u_1, u_2)$ a function of the type $\Omega_2 f(u_1, u_2)$ the formula still holds provided on the right hand side in the first line Ω_1 is replaced by Ω_2 and in the second line $\frac{\partial M_2(\pi)}{\partial u_2}$ is replaced by $-\frac{\partial J_2(\pi)}{\partial u_2}$. Collecting these facts one can easily write down a closed expression for the full $(n + 3)$ -loop correlator. We shall refrain from doing so. That the stated form of the $(n + 3)$ -loop correlator is indeed correct can be proven by induction (generalizing the idea of reference [11]) using the following identities

$$[\hat{P}, \hat{Q}] = 0 \tag{4.19}$$

and

$$\begin{aligned}
\left[\frac{\partial}{\partial V(\pi)}, \hat{P}^n \hat{Q}^m \Omega_1 \right] &= \hat{P}^n \hat{Q}^{m+1} \Omega_1 M_2(\pi) + \hat{P}^{n+1} \hat{Q}^m \Omega_1 J_2(\pi) \\
&\quad - M_2(\pi) \hat{P}^n \hat{Q}^{m+1} \Omega_1 - J_2(\pi) \hat{P}^{n+1} \hat{Q}^m \Omega_1 \\
&\quad + \hat{P}^n \hat{Q}^m \left(u_1 \Omega_1^2 + \Omega_2^2 \right) \frac{\partial M_2(\pi)}{\partial u_2},
\end{aligned} \tag{4.20}$$

$$\begin{aligned}
\left[\frac{\partial}{\partial V(\pi)}, \hat{P}^n \hat{Q}^m \Omega_2 \right] &= \hat{P}^n \hat{Q}^{m+1} \Omega_2 M_2(\pi) + \hat{P}^{n+1} \hat{Q}^m \Omega_2 J_2(\pi) \\
&\quad - M_2(\pi) \hat{P}^n \hat{Q}^{m+1} \Omega_2 - J_2(\pi) \hat{P}^{n+1} \hat{Q}^m \Omega_2 \\
&\quad - \hat{P}^n \hat{Q}^m \left(\Omega_2^2 + u_1 \Omega_1^2 \right) \frac{\partial J_2(\pi)}{\partial u_2},
\end{aligned} \tag{4.21}$$

$$\begin{aligned}
\left[\frac{\partial}{\partial V(\pi)}, \hat{P}^n \hat{Q}^m \left(u_1 \Omega_1^2 + \Omega_2^2 \right) \right] &= \\
&\quad \hat{P}^n \hat{Q}^{m+1} \left(u_1 \Omega_1^2 + \Omega_2^2 \right) M_2(\pi) + \hat{P}^{n+1} \hat{Q}^m \left(u_1 \Omega_1^2 + \Omega_2^2 \right) J_2(\pi) \\
&\quad - M_2(\pi) \hat{P}^n \hat{Q}^{m+1} \left(u_1 \Omega_1^2 + \Omega_2^2 \right) - J_2(\pi) \hat{P}^{n+1} \hat{Q}^m \left(u_1 \Omega_1^2 + \Omega_2^2 \right)
\end{aligned} \tag{4.22}$$

The three last relations themselves can likewise be proven by induction. The only non-standard part is to realize that the following equations hold

$$\frac{\partial \Omega_1}{\partial V(\pi)} = \Omega_1 \left(\hat{P} J_2(\pi) \right) + \Omega_2 \left(\hat{P} M_2(\pi) \right), \tag{4.23}$$

$$\begin{aligned}
\frac{\partial \Omega_2}{\partial V(\pi)} &= (-u_1) \Omega_1 \left(\hat{P} M_2(\pi) \right) + \Omega_2 \left(\hat{P} J_2(\pi) \right) \\
&= \Omega_2 \left(\hat{Q} M_2(\pi) \right) + \Omega_1 \left(\hat{Q} J_2(\pi) \right).
\end{aligned} \tag{4.24}$$

It is easy to see that among the contributions to the n -loop correlator ($n > 3$) we have terms of the same type as those constituting the 3-loop correlator, namely terms consisting of the n -point vertices of the gravitational primary fields saturated by propagators, $\mathcal{P}^0(\pi)$ and $\mathcal{P}^1(\pi)$. This follows from the following observation

$$\begin{aligned}
\hat{P}f &= \left(\frac{3}{2}c \right) \frac{df}{dt_{0,0}} \iff \frac{\partial f}{\partial t_{0,0}} = 0, \\
\hat{Q}f &= \left(\frac{3}{4}\tilde{c} \right) \frac{df}{dt_{1,0}} \iff \frac{\partial f}{\partial t_{1,0}} = 0
\end{aligned}$$

and the fact that neither Ω_1 nor Ω_2 has any explicit dependence on θ_0 or $\tilde{\theta}_0$ (cf. to (3.15) and (3.16)). We would of course expect terms of the type just mentioned to be present for any series of rational matter fields coupled to gravity. The terms which are not of

this type all contain products of (at most $(n+1)$) m -point vertices with $3 \leq m \leq n-1$. It would be interesting to disentangle the π -dependent factors in these terms to obtain an interpretation of these in terms of internal and external propagators in the spirit of the theory of strings with discrete target space [13]. This would provide us with an expression which could be immediately generalized to the case of the generic rational matter field.

Let us close this section by remarking that the formula (4.18) is a little more involved than one could have hoped for knowing the corresponding formula for the case of minimal models with $(p, q) = (2, 2m-1)$ coupled to gravity [11, 12]. In the latter case the expression for the $(n+3)$ -loop correlator consists of only one term with a structure similar to that of the first term on the right hand side of (4.18). However, for a model with the presence of two operators with different dimensions we must accept a less simple result.

5 Outlook

Using the moment description of the generic 1-matrix model [9] it was proven that the free energy of the Kontsevich model was exactly equal to that of the generic 1-matrix model with all subleading singularities subtracted [15]. It would be interesting to establish a similar correspondence between the two matrix model and the generalized Kontsevich models. The $p = 3$ version, that we have considered here, we would expect to have a singularity structure describing the leading singularities of a two-matrix model where one matrix potential is cubic and the other one arbitrary. This is of course in accordance with the fact that the coupling to gravity of all rational matter fields of the type $(3, 3m-1)$, $(3, 3m-2)$ can be described by a two-matrix model of the type mentioned [5]. The correspondence is furthermore outlined by the following fact. For a two-matrix model with one potential cubic the loop equation giving the 1-loop correlator of the matrix with the arbitrary potential (an algebraic equation of degree 3 [16]) has exactly the same structure as the matrix Airy equation satisfied by $Z_3^N(\Lambda)$ [7]. Likewise we would expect the $p = n$ version of the Kontsevich model to give exactly the leading singular behaviour of a two matrix model with one potential of degree n and the other one arbitrary. A moment description of the two-matrix model has not yet been found but should certainly exist. In the light of the discussion above a reasonable strategy for finding such a description would be to start by considering a two-matrix model with one potential cubic. One would expect more than two series of moments to be necessary in analogy with the 1-matrix case where one set of moments was sufficient for the double scaling limit but two sets were needed away from this limit. Likewise on the basis of the experience from the 1-matrix model one might expect

complications to occur at genus zero. Finding the appropriate moment description for the two matrix model would, however, provide us with the exact correspondence between the matrix model coupling constants and the continuum time variables used in the context of topological gravity which again would allow us to understand the connection between the matrix model observables and the continuum scaling operators.

Furthermore it would be interesting to elaborate on the correspondence between the present approach and the approach of strings with discrete target space [13] in order to obtain a geometrically more appealing version of (4.18) as well as a generalization thereof to arbitrary (p, q) rational matter fields coupled to gravity.

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References

- [1] M. Douglas and S. Shenker, Nucl. Phys. B335 (1990) 635
E. Brezin and V. Kazakov, Phys. Lett. B236 (1990) 144
D. Gross and A.A. Migdal, Phys. Rev. Lett. 64 (1990) 127
- [2] V. Kazakov, Mod. Phys. Lett. A4 (1989) 2125
M. Staudacher, Nucl. Phys. B336 (1990) 349
- [3] M. Kontsevich, Funk. Anal. & Prilozh 25 (1991) 50; Comm. Math. Phys. 147 (1992) 25
- [4] E. Witten, *Algebraic geometry associated with matrix models of two dimensional gravity*, IASSNS-HEP-91/74
- [5] M. Douglas, Phys. Lett. B238 (1990) 176
J.-M. Daul, V.A. Kazakov and I.K. Kostov, Nucl. Phys. B409 (1993) 311
- [6] M. Adler and P. Van Moerbeke, Comm. Math. Phys. 147 (1992) 25
- [7] C. Itzykson and J.-B. Zuber, J. Math. Phys. A7 (1992) 5661
- [8] S. Kharchev, A. Marshakov, A. Mironov, A. Morozov and A. Zabrodin, Nucl. Phys. B380 (1992) 181
- [9] J. Ambjørn, L. Chekhov, C.F. Kristjansen and Yu. Makeenko, Nucl. Phys. B404 (1993) 127

- [10] T. Eguchi, Y. Yamada and S.-K. Yang, *On the Genus expansion in the Topological String Theory*, UTHEP-275
- [11] J. Ambjørn, J. Jurkiewicz and Yu. Makeenko, Phys. Lett. B251 (1990) 517.
- [12] G. Moore, N. Seiberg and M. Staudacher, Nucl. Phys. B362 (1991) 665
- [13] I.K. Kostov, Phys. Lett. B266 (1991) 41, Phys. Lett. B266 (1991) 42; Phys. Lett. B266 (1991) 317, Nucl. Phys. B376 (1992) 539
- [14] E.R. Hansen, *A table of Series and Products*, Prentice-Hall, 1975
- [15] J. Ambjørn and C. Kristjansen, Mod. Phys. Lett. A8 (1993) 2875
- [16] E. Gava and K.S. Narain, Phys. Lett. B263 (1991) 213
J. Alfaro, Phys. Rev. D47 (1993) 4714
M. Staudacher, Phys. Lett. B305 (1993) 332